Elliptical beams

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Abstract: A very general beam solution of the paraxial wave equation in elliptic cylindrical coordinates is presented. We call such a field an elliptic beam (EB). The complex amplitude of the EB is described by either the generalized Ince functions or the Whittaker-Hill functions and is characterized by four parameters that are complex in the most general situation. The propagation through complex ABCD optical systems and the conditions for square integrability are studied in detail. Special cases of the EB are the standard, elegant, and generalized Ince-Gauss beams, Mathieu-Gauss beams, among others.

OCIS codes: (050.1940) Diffraction; (260.1960) Diffraction theory; (350.500) Propagation.

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1. Introduction

In recent years, increasing interest has been paid to explore the properties and applications of paraxial optical beams with elliptic geometry. For example, the existence of the standard Ince-Gaussian (sIG) [1, 2], elegant IG (eIG) [3], and generalized IG (gIG) [4] beams, which are solutions of the paraxial wave equation (PWE) in elliptic coordinates, was theoretically demonstrated. Standard IG beams constitute the third complete family of transverse eigenmodes of stable resonators, and have been experimentally generated with diode-pumped Nd:YVO_4 lasers [5], laser-diode-pumped microchip solid-state lasers [6, 7, 8], and liquid crystal displays [9]. Other family of solutions of the PWE with elliptic geometry is given by the Mathieu-Gauss (MG) beams [10], that constitute one of the simplest physical realizations of the exact nondiffracting Mathieu beams, and their propagation properties have been studied in free space [11] and through ABCD optical systems [12].

In this paper, we introduce a new and very general beam solution of the PWE in elliptic coordinates. These new solutions are called elliptical beams (EB) and are characterized by four parameters that are complex in the most general situation. The possibility of choosing arbitrary complex values for the beam parameters allows us to obtain novel and meaningful beam structures, that, to our knowledge, have not yet been reported in the literature. For special values of the beam parameters, the EB reduce to the known sIG, eIG, gIG, and MG beams. The propagation of the EB (and consequently of its known special cases) through complex ABCD optical systems and the conditions for square integrability of the EBs are also discussed in detail. The recently introduced Cartesian [13, 14] and circular beams [15] correspond to limiting cases of the new EBs when the ellipticity parameter tends to infinity or to zero, respectively.

2. Elliptic beams in free space

Consider the free-space propagation of a beam of wavenumber k along the positive z axis of a coordinate system $(x,y,z) = (\mathbf{r},z) = (r\cos\theta, r\sin\theta, z)$. To obtain a general solution of the PWE

$$(\partial_x^2 + \partial_y^2 + 2ik\partial_z)U(\mathbf{r},z) = 0, \tag{1}$$

for the slowly varying complex envelope $U(\mathbf{r},z)$ we propose the ansatz $U(\mathbf{r},z) = Z(z)F(u,v)\mathrm{GB}(r,q)$, where $\boldsymbol{\rho} \equiv (u,v) \equiv \mathbf{r}/\chi(z)$ are scaled Cartesian coordinates, $\chi(z)$ is a z-dependent scaling factor to be determined, $\mathrm{GB}(r,q)$ is the fundamental Gaussian beam

$$\mathrm{GB}(r,q) = [q_0/q(z)]\ \exp\left[ikr^2/2q(z)\right], \tag{2}$$

where $q(z) = z + q_0$ is the standard complex beam parameter for free-space propagation [16], and q_0 is the beam parameter at $z = 0$. Substitution of the ansatz into Eq. (1) leads to the following differential equations for $F(u,v)$, $\chi(z)$, and $Z(z)$:

$$\left[\overline{\nabla}^2 - \boldsymbol{\rho}\cdot\overline{\nabla} + i\gamma - 1\right]F = 0, \tag{3a}$$

$$\partial_z\left(\chi^2/q^2\right) = -i/kq^2, \tag{3b}$$

$$\partial_z Z/Z = (\gamma + i)/2k\chi^2, \tag{3c}$$

with $\overline{\nabla} = (\partial_u, \partial_v)$ and γ being a separation constant. The solution of Eq. (3b) yields

$$1/\chi^2(z) = ik\left[1/\widetilde{q}(z) - 1/q(z)\right], \tag{4}$$

where $\widetilde{q}(z) = z + \widetilde{q}_0$ is a second complex beam parameter and $\widetilde{q}_0$ is an integration constant. The solution of Eq. (3c) gives $Z(z) = (\widetilde{q}q_0/q\widetilde{q}_0)^{i\gamma/2-1/2}$, such that $Z(0) = 1$.

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To solve Eq. (3a) we will assume a solution with the separable form $F(\xi,\eta) = E(\xi)N(\eta)$. In a transverse z plane, we define the elliptic coordinates (ξ,η) as

$$u = x/\chi(z) = \sqrt{2\varepsilon}\cosh\xi\cos\eta, \qquad v = y/\chi(z) = \sqrt{2\varepsilon}\sinh\xi\sin\eta, \tag{5}$$

where ε denotes the ellipticity parameter of the beam. Note that the elliptic coordinates (ξ,η) are complex in order to satisfy the requirement that the Cartesian coordinates (x,y) remain real in the entire space.

The separation of Eq. (3a) leads to the Ince equations with complex coefficients [17]

$$\partial_{\eta\eta}N + \varepsilon\sin(2\eta)\,\partial_\eta N + [\mu - \varepsilon(i\gamma-1)\cos(2\eta)]N = 0, \tag{6a}$$

$$\partial_{\xi\xi}E - \varepsilon\sinh(2\xi)\,\partial_\xi E - [\mu - \varepsilon(i\gamma-1)\cosh(2\xi)]E = 0, \tag{6b}$$

with μ being the separation constant. The solutions of Eq. (6a) are given by the generalized Ince functions $\mathrm{C}_\gamma^{\sigma,m}(\eta,\varepsilon)$ of argument $\eta\in\mathbb{C}$, degree $m = 0,1,2,...$, parameters $\gamma\in\mathbb{C}$ and $\varepsilon\in\mathbb{C}$, and parity $\sigma = \{e,o\}$. Notice that Eq. (6b) can be derived from Eq. (6a) by writing $i\xi$ for η and vice versa. Therefore solutions to Eq. (6b) can be obtained by analytic continuation of the solutions of the Ince equation.

By collecting the partial results and rearranging terms we obtain the general expression of the EBs,

$$\mathrm{EB}_\gamma^{\sigma,m}(\mathbf{r};\varepsilon,q,\widetilde{q}) = (\widetilde{q}q_0/q\widetilde{q}_0)^{(i\gamma-1)/2}\ \mathrm{C}_\gamma^{\sigma,m}(i\xi,\varepsilon)\mathrm{C}_\gamma^{\sigma,m}(\eta,\varepsilon)\mathrm{GB}(r,q). \tag{7a}$$

Equation (7a) is an exact solution of the PWE in free space [Eq. (1)] and represents the first important result of this paper. By rearranging terms we can rewrite the elliptic beam $\mathrm{EB}_\gamma^{\sigma,m}$ in the following symmetric form:

$$\mathrm{EB}_\gamma^{\sigma,m}(\mathbf{r};\varepsilon,q,\widetilde{q}) = (\widetilde{q}q_0/q\widetilde{q}_0)^{i\gamma/2}\ \mathrm{hc}_\gamma^{\sigma,m}(i\xi,\varepsilon)\,\mathrm{hc}_\gamma^{\sigma,m}(\eta,\varepsilon)\sqrt{\mathrm{GB}(r,q)\ \mathrm{GB}(r,\widetilde{q})}, \tag{7b}$$

where $\mathrm{hc}_\gamma^{\sigma,m}(\eta,\varepsilon) = \exp\left(-\frac{1}{4}\varepsilon\cos 2\eta\right)\mathrm{C}_\gamma^{\sigma,m}(\eta,\varepsilon)$ is the mth-order solution of the Whittaker-Hill equation (i.e. Hill equation with three terms) [17]:

$$\left[\mathrm{d}_{\eta\eta} + \mu - \left(\varepsilon^2/8\right) - i\gamma\varepsilon\cos 2\eta + (\varepsilon^2/8)\cos 4\eta\right]\mathrm{hc}_\gamma^{\sigma,m}(\eta,\varepsilon) = 0. \tag{8}$$

At any transverse z plane, the shape of the even and odd EBs is characterized by five parameters. First, γ and m play the role of a complex continuous radial order and an integer angular mode number, respectively. The ellipticity of the EB is determined by the arbitrarily complex parameter ε. Finally, the parameters $q(z)$ and $\widetilde{q}(z)$ control the properties of two independent complex Gaussian apodizations and are strongly related to the square integrability conditions and the physical size of the beam.

Under free space propagation along a distance z, the field parameters transform according to the translation property $(\gamma,m,\varepsilon,q_0,\widetilde{q}_0)\longmapsto(\gamma,\varepsilon,m,q_0+z,\widetilde{q}_0+z)$, retaining the functional form of the wave field. The parameters γ, m, and ε can be associated to conserved quantities, therefore their magnitudes keep constant as the beam propagates, even through arbitrary ABCD systems. On the other hand, the parameters $q(z)$ and $\widetilde{q}(z)$ describe local properties of the beam and therefore their magnitudes depend on the observation z plane. The complex focus of the elliptic coordinate system that, at the plane z, is given by $f(z) = \sqrt{(2\varepsilon/ik)q\widetilde{q}/(q-\widetilde{q})}$ depends on the observation plane as well.

It is important to remark that the EB [Eq. (7)] is invariant under the parameter transformation $(\gamma,\varepsilon,q,\widetilde{q})\Leftrightarrow(-\gamma,-\varepsilon,\widetilde{q},q)$.

From a physical point of view, it is important to identify the range of values of $(\gamma,\varepsilon,q,\widetilde{q})$ for which the EBs transport finite power, i.e., for which are square integrable across the whole

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transverse plane. First, the case $\gamma = -i(p+1)$ with $p = 0,1,2,...$ leads to sIG, eIG, gLG, and MG for which the square integrability is ensured by setting $\mathrm{Im}(1/q) > 0$.

For arbitrary $\gamma \neq -i(p+1)$, consider an integrability plane whose axes are associated with $\mathrm{Im}(1/q)$ and $\mathrm{Im}(1/\widetilde{q})$. Each point on the plane is associated with the pair of values $[\mathrm{Im}(1/q), \mathrm{Im}(1/\widetilde{q})]$ that the beam acquires at a given transverse z plane. The beam (a) is square integrable if the point falls within the first quadrant of the integrability plane, and (b) diverges as $r \to \infty$ if the point falls within the second, third, or fourth quadrants. If the point falls on the positive $\mathrm{Im}(1/q)$-axis then the square integrability depends on γ as follows: the beam (i) is square integrable if $\mathrm{Im}\,\gamma < 0$; (ii) tends to zero as $|r| \to \infty$ but it is not square integrable if $\mathrm{Im}\,\gamma \in [0,1)$; (iii) tends to a constant nonzero value as $|r| \to \infty$ if $\mathrm{Im}\,\gamma = 1$; (iv) diverges if $\mathrm{Im}\,\gamma > 1$. Finally, if the point falls on the positive $\mathrm{Im}(1/\widetilde{q})$-axis then we have the same cases as in the $\mathrm{Im}(1/\widetilde{q})$-axis but with γ replaced by $-\gamma$ and ε by $-\varepsilon$.

3. Propagation through ABCD systems

Equation (7) describes the propagation of the EBs in free space. We will now study the propagation of the EBs through a ABCD optical systems. The standard approach to calculate this propagation consist on solving the Collins diffraction integral for an input field of the form (7), however the corresponding integral involving the complex Ince functions is unknown in the mathematics literature. To save this drawback, we have applied the theory developed recently by Bandres and Guizar-Sicarios [18], where they showed that, if the free-space propagation of a paraxial beam is known, then the propagation of the beam through an ABCD system can be straightforwardly obtained by applying the symmetries of the PWE. Applying this result to the EBs, we get that if an EB with parameters $(\gamma, \varepsilon, q_0, \widetilde{q}_0)$ at the input plane $(\mathbf{r}_0, z_0)$ of a paraxial system has the general form

$$\Psi_0(\mathbf{r}_0; \varepsilon, q_0, \widetilde{q}_0) = \mathrm{C}_\gamma^{\sigma,m}(i\xi_0, \varepsilon)\mathrm{C}_\gamma^{\sigma,m}(\eta_0, \varepsilon)\exp\left(ikr_0^2/2q_0\right), \tag{9}$$

then the field at the output plane $(\mathbf{r}_1, z_1)$ after the ABCD system is given by

$$\Psi_1(\mathbf{r}_1; \varepsilon, q_1, \widetilde{q}_1) = \left(\frac{A + B/\widetilde{q}_0}{A + B/q_0}\right)^{i\gamma/2-1/2} \mathrm{C}_\gamma^{\sigma,m}(i\xi_1, \varepsilon)\mathrm{C}_\gamma^{\sigma,m}(\eta_1, \varepsilon)\mathrm{GB}(r_1, q_1), \tag{10}$$

where the elliptic coordinates at the input ($j = 0$) and output ($j = 1$) planes read as

$$x_j = \sqrt{2\varepsilon}\chi_j(z)\cosh(\xi_j)\cos(\eta_j), \qquad y_j = \sqrt{2\varepsilon}\chi_j(z)\sinh(\xi_j)\sin(\eta_j), \tag{11}$$

and $\mathrm{GB}(r_1, q_1) = (A + B/q_0)^{-1}\exp\left(ikr_1^2/2q_1\right)$ is the output field of a Gaussian beam with input parameter q_0 traveling axially through the ABCD system and the transformation laws for the beam parameters from the input plane z_0 to the output plane z_1 are

$$q_1 = (Aq_0 + B)/(Cq_0 + D), \qquad \widetilde{q}_1 = (A\widetilde{q}_0 + B)/(C\widetilde{q}_0 + D). \tag{12}$$

Equation (10) is the second important result of this paper. It allows an EB to be traced through an arbitrary real or complex ABCD optical system. The output function for a EB with input parameters $(\gamma, \varepsilon, q_0, \widetilde{q}_0)$ propagating through a paraxial system has the same functional form as the input function but with new values for the parameters $(\gamma, \varepsilon, q_1, \widetilde{q}_1)$. As expected, Eq. (10) reduces to Eq. (7) for the case of free-space propagation, i.e. when $ABCD = [1, z; 0, 1]$.

At both the input and output planes, the scaling parameter $\chi(z)$ is determined from the complex beam parameters q and $\widetilde{q}$ according to Eq. (4). Using Eqs. (12) it is easy to show that χ_1^2 may also be written in the equivalent forms

$$\chi_1^2 = \chi_0^2(A + B/q_0)(A + B/\widetilde{q}_0) = (A + B/q_0)^2\left[\chi_0^2 - iB/k(A + B/q_0)\right], \tag{13}$$

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Table 1. Special cases of the EB [Eq. (7)] at $z = 0$.

Special cases	γ	ε	q_0	$\widetilde{q}_0$	Comments
sIG_p^m eIG_p^m gIG_p^m	$-i(p+1)$	$\varepsilon > 0$	q_0	q_0^* ∞ $\widetilde{q}_0$	$p = 0, 1, 2, \ldots,$ $(-1)^{p-m} = 1$
Mathieu-Gauss [10]	$-i(p+1)$ $p \to \infty$	$\varepsilon \to 0$	q_0	$\left(q_0^{-1} - i\frac{k_t^2/k}{p-m}\right)^{-1}$	$p\varepsilon \to 2\Omega$ = finite
x^α GB	γ	ε	q_0	q_0	$\alpha = \left[\frac{1+(-1)^m}{2}\right]$
Cartesian beam [13] Circular beam [15] Elliptic vortex	γ γ $-i$	∞ 0 ε	q_0 q_0 q_0	$\widetilde{q}_0$ $\widetilde{q}_0$ 0	

The second expression in Eq. (13) can be recognized as the transformation law for the "complex spot size" of the generalized Hermite-Gaussian beams studied by Siegman (p. 798 of [16]). This new parameterization that we have introduced to describe the EB in terms of the two independent complex beams parameters q and $\widetilde{q}$ is more convenient for the following reasons: first, the transformation law for $\widetilde{q}$ takes exactly the same and well-known bilinear relation as that for the complex beam parameter q [Eqs. (12)]. Second, the invariance of the EBs under the transformation $(\gamma, \varepsilon, q, \widetilde{q}) \Leftrightarrow (-\gamma, -\varepsilon, \widetilde{q}, q)$ is clearly evident. Finally, the conditions for square integrability of the EB are expressed in a simpler and symmetrical form.

4. Examples and special cases

Particular values of the EB parameters lead to known solutions of the PWE that have been well studied in the literature. Table 1 shows a list of these special cases and the corresponding values of $(\gamma, \varepsilon, q_0, \widetilde{q}_0)$ at the initial plane $z_0 = 0$. Notice that the relations between the known special cases, that have been not clear in the past, now become evident due to our parametrization of the EBs. For example, the limiting case of $\mathrm{EB}^{0,m}_{-i(p+1)}(\mathbf{r}; \varepsilon, q_0, \widetilde{q}_0)$ when $\varepsilon \to 0$, $p \to \infty$, while the product $p\varepsilon \to 2\Omega$, corresponds to the Mathieu-Gauss beams $\mathrm{Je}_m(\xi, \Omega)\mathrm{ce}_m(\eta, \Omega)\exp\left(-ikr^2/2q_{\mathrm{MG}}\right)$ with transverse wave number $k_t = [i(p-m)k(1/\widetilde{q}_0 - 1/q_0)]^{1/2}$ and complex beam parameter $1/q_{\mathrm{MG}} = 1/2q_0 + 1/2\widetilde{q}_0$, where Je_m and ce_m are the mth-order radial and angular Mathieu functions [10] of parameter Ω.

Note in Table 1 that the already known special cases of the EBs correspond to purely imaginary negative integer values of γ and purely real positive values of ε. Arbitrary complex values of γ and ε lead to new beam structures that, to our best knowledge, have not yet been reported in the optics literature. Circular beams with azimuthal angular dependence $\exp(im\theta)$ have a phase that rotates circularly around the propagation axis [15]. In a similar way, from the stationary EBs described by Eq. (7), it is possible to construct helical EB (HEB) beams of the form $\mathrm{HEB}^{\pm,m}_\gamma = \mathrm{EB}^{0,m}_\gamma \pm i\mathrm{EB}^{1,m}_\gamma$, but whose phase exhibits now an arrangement of optical vortices at positions depending of the beam parameters. For example, the penultimate row of Fig. 1 shows the intensity and phase of a positive HEB at the planes $z = 0$ and $z = 1.5z_R$ for $m = 1$. Because in the limit $\varepsilon \to 0$ the HEBs reduces to helical circular beams, we can see that under a small perturbation of the elipticity around this limit the axial higher-order vortex of the helical circular beam unfolds into a straight row of unit-strength vortices [19].

The combination of values in the bottom row of Table 1 produces a new structure that we call elliptic vortex beam (EVB) whose phase at $z = 0$ exhibits an axial vortex of topological charge m, as shown in the last row of Fig. 1. Contrary to the axially symmetric higher-order beams, like

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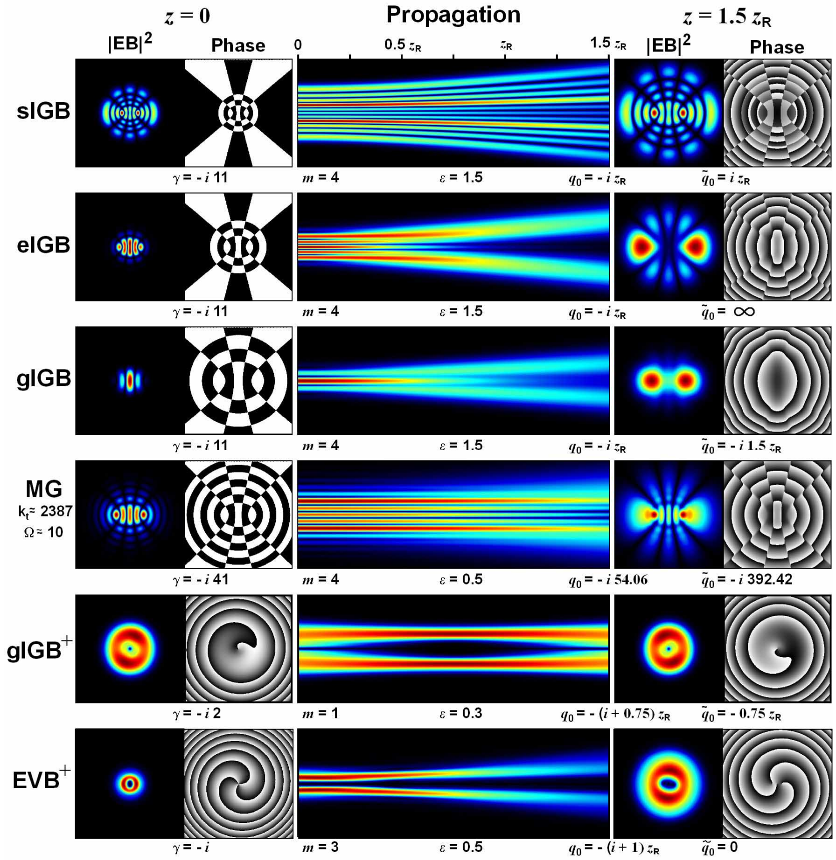

Fig. 1. Intensity and phase patterns at the planes $z = 0$ and $z = 1.5z_R$ and along the plane $(x, 0 \leq z \leq 1.5\, z_R)$ for several special cases of the EBs.

Laguerre-Gauss and Bessel beams, the initial m-th order phase singularity of the helical EVB unfolds into m unit-strength vortices outside of the plane $z = 0$. To our best knowledge, the EVB has not been yet reported in the optics literature, and their physical properties are currently under study by the authors. The general expression Eq. (10) can be applied to propagate all special cases reported in Table 1 through ABCD optical systems.

Finally, we remark that the EBs discussed in this paper differ from the Hermite-Laguerre modes [20] in the sense that the EBs result from the exact separation of the PWE in elliptic coordinates instead of coherent superposition of known beams with rectangular or circular geometry. This mathematical construction allows to establish complete families of beam solutions with elliptical symmetry whose mathematical form is expressed in close-form. The EBs can be expanded in terms of other complete families of paraxial beams like the Hermite-Gauss or Laguerre-Gauss beams and vice versa.

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